UDC 534.2:532

# THE PATTERN OF SOUND WAVE SCATTERING ON A CONVEX SHELL

PMM Vol. 43, No. 2, 1979, pp. 314-320

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(Received January 6, 1978)

The high-frequency asymptotics of the scattering pattern of a plane sound wave is derived by analyzing the diffraction field near a convex shell. Formula for scattering pattern is obtained in the form of an integral over the shell surface which depends on pressure and its normal derivative. Asymptotics of the integral for large angles between the direction of the incident field and the line of observation is determined using the method of stationary phase. Asymptotics is investigated in the case of small angles.

1. The integral formula for the pattern of scattering. Let a plane high-frequency sound wave $P_+ = \exp\{-i[\omega t - k(\mathbf{r}, \boldsymbol{\theta}_0)]\}$ impinge in the direction $\boldsymbol{\theta}_0$ on a thin conves shell placed in a fluid. Pressure outside the shell conforms to the Helmholtz equation. The dependence of pressure fields on time defined by the coefficient $\exp(-i\omega t)$ is not subsequently indicated and this coefficient is omitted. Oscillations of the shell are defined by the theory of thin shells [1] supplemented by the condition of equality of normal components of the shell velocity and of the fluid on the shell.

Since the field $(P_-)$ scattered by the shell satisfies the Helmholtz equation and the conditions of radiation at infinity, hence in conformity with Green's formula

$$P_-(r\boldsymbol{\theta}) = \frac{1}{4\pi}\int\limits_S \left\{\frac{\partial}{\partial n}\left(\frac{e^{ik\rho}}{\rho}\right)P_- - \frac{e^{ik\rho}}{\rho}\frac{\partial P_-}{\partial n}\right\} ds \tag{1.1}$$

where $S$ is the surface of the shell, $k$ is the wave number, $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}$ are unit vectors in the direction of the incident field and of observation, respectively, $r$ and $\rho$ represent the distances of the observation point from the coordinate origin and a point on the shell surface, and $n$ is an outward normal to the shell. We assume the coordinate origin to be inside the shell. In the case of considerable distances to the observation point

$$\frac{e^{ik\rho}}{\rho} = \frac{\exp\{ik[r - (\boldsymbol{\theta}, \mathbf{r})]\}}{r}\left(1 + O\left(\frac{\mu^2}{kr}\right)\right) \tag{1.2}$$

$$\frac{\partial}{\partial n}\left(\frac{e^{ik\rho}}{\rho}\right) = -ik(\boldsymbol{\theta}, \mathbf{n})\frac{\exp\{ik[r - (\boldsymbol{\theta}, \mathbf{r})]\}}{r}\left(1 + O\left(\frac{\mu^2}{kr}\right)\right)$$

$$\mu = D / \lambda$$

where $D$ is the maximum distance of points of the shell from the coordinate origin and $\lambda$ is the wave length. Since owing to the high-frequency of the impinging field $\mu$ is large, hence, if the correction in (1.2) is to be small, it is necessary that condition $kr \gg \mu^2$ is satisfied. For large $r$ from (1.1) and (1.2) we obtain

$$P_{-}(r\theta) = e^{ikr}r^{-1} f(\theta) [1 + O(\mu^2 / (kr))]$$

For function $f(\theta)$ which is called the "pattern of scattering" the following formula is valid:

$$f(\theta) = -\frac{1}{4\pi}\int\limits_S \left\{\frac{\partial P_-}{\partial n} + ik(\theta, \mathbf{n}) P_-\right\} \exp\{-ik(\theta, \mathbf{r})\} ds$$

Let us determine $P_-$ close to the shell. Derivation of the diffraction field near the shell when $kh \ll 1$ ($h$ is the shell thickness) is given in [2] (see, also, [3] ). Formulas for any arbitrary $kh$ can be obtained in the same manner. Formulas for $P_-$ are different in the illuminated, shadow, and penumbra regions. The penumbra region is understood to be the region of width $\sim \mu^{-1/3+\varepsilon}$ surrounding the contour $L$, which is the boundary that separates the region of the geometric shadow from the illuminated region of the shell.

In the illuminated region outside the penumbra

$$P_- \sim -[(1 - A) / (1 + A)] \exp\{ik\tau_-\}$$

$$A = \frac{i(\theta_0, \mathbf{n})}{\rho_2 a^2}\left[\rho_1 kha^2 - \frac{E(kh)^3}{12(1-\sigma^2)}(\nabla_2 x)^4 - \frac{\beta}{\Lambda}\right]$$

$$\beta = Eh\left\{\frac{1}{h_1^2 R_2}\left(\frac{\partial x}{\partial x_1}\right)^2 + \frac{1}{h_2^2 R_1}\left(\frac{\partial x}{\partial x_2}\right)^2\right\}^2$$

$$\Lambda = k\{(\nabla_2 x)^2 - \rho_1 a^2 (1 - \sigma^2) / E\}\{(\nabla_2 x)^2 - 2\rho_1 a^2 (1 + \sigma) / E\}$$

where $\tau_-$ is the eikonal of the reflected field $x_i$, $h_i$, and $R_i$ are coordinates of curvature lines on the shell surface, the Lamé coefficients, and the curvature radii respectively, $\rho_1$ and $\rho_2$ are the densities of the shell and fluid outside it, $E$ and $\sigma$ are the Young modulus and the Poisson coefficient, $a$ is the speed of sound in the fluid, $x = (\theta_0, \mathbf{r})$, and $\nabla_2$ is the Hamiltonian operator in coordinates $(x_1, x_2)$.

Let us define the system of coordinates $(y, z)$ on the surface of the shell. The $z$ -coordinate defines the position of a point on contour $L$, and $y$ is represents the length of the geodesic, accurate to the sign, emerging from contour $L$ in the direction $\theta_0$, and has the plus sign in the shade and the minus sign in the illuminated zone of the shell

$$s = (k / (2R^2))^{1/3} y, \quad \nu = (2k^2 / R)^{1/3} n$$

where $R$ is the geodesic curvature radius. Then in the penumbra region

$$P_- \sim \frac{1}{\sqrt{\pi}} \exp\{ik\tau(y, z)\} \int\limits_l e^{is\zeta} G(\zeta) w_1(\zeta - \nu) d\zeta$$

$$l = l_1 \cup l_2 \cup l_3 = l_-(0, 2\pi/3) \cup l_-(0, 4\pi/3) \cup l_+(0, 0)$$

$$\tau(y, z) = x(0, z) + y$$

$$G(\zeta) = -[v(\zeta) + Cv'(\zeta)] / [w_1(\zeta) + Cw_1'(\zeta)]$$

$$C = \left(\frac{2}{kR}\right)^{1/3}\left\{\frac{\rho_1}{\rho_2} kh - \frac{E(kh)^3}{12\rho_2 a^2(1-\sigma^2)} - \frac{\beta_1}{\Lambda_1}\right\}$$

$$\beta_1 = E\,(kh)(R^{-1} - R_1^{-1} - R_2^{-1})^2$$
$$\Lambda_1 = \rho_2\,(ka)^2[1 - \rho_1 a^2\,(1-\sigma^2)\,/\,E][1 - 2\rho_1 a^2\,(1+\sigma)\,/\,E]$$

where $v$ and $w_1$ are Airy functions, $l_+\,(b,\,\varphi)$ is the ray in the complex plane which emerges from point $b$ at angle $\varphi$ to the real axis, and $l_-\,(b,\,\varphi)$ is the same contour but traversed in the opposite direction.

In the shade $P_- = -P_+$ with an accuracy within terms exponentially small with respect to $\mu$ .

Let $\{\chi_i\}_{i=1}^3$ be the subdivision of unity on the shell surface, where the carrier $\chi_1$ lies in the illuminated region, carrier $\chi_2$ in the penumbra region, and carrier $\chi_3$ in the shade region. The pattern of scatter can now be presented in the form of three integrals

$$f(\theta) = \sum_{m=1}^{3} f_m(\theta)$$

For $f_m\,(\theta)$ $(m = 1,\ 2,\ 3)$ we obtain the following formulas:

$$f_1(\theta) = \frac{ik}{4\pi}\int_S \chi_1(\theta-\theta_0,\,\mathbf{n})\,\frac{1-A}{1+A}\exp\{-ik(\theta-\theta_0,\,\mathbf{r})\}\,ds \tag{1.3}$$

$$f_2(\theta) = \frac{-\pi^{-3/2}}{4}\,ik\int_S \chi_2\exp\{ik(\tau-(\theta,\,\mathbf{r}))\}\int_l e^{is\zeta}G(\zeta)\,\times$$
$$\left[(\theta,\,\mathbf{n})\,w_1(\zeta) + i\left(\frac{2}{kR}\right)^{1/3} w_1'(\zeta)\right]d\zeta\,ds$$
$$f_3(\theta) = \frac{ik}{4\pi}\int_S \chi_3(\theta+\theta_0,\,\mathbf{n})\exp\{-ik(\theta-\theta_0,\,\mathbf{r})\}\,ds$$

2. Asymptotics of the pattern of scatter for $|\theta - \theta_0| \gg \mu^{-1/3+\varepsilon}$. Let us determine the disposition of critical points of integrals in (1.3). For integrals $f_1\,(\theta)$ and $f_3\,(\theta)$ they are determined by conditions

$$(\theta-\theta_0,\ \partial\mathbf{r}\,/\,\partial x_i) = 0, \quad i = 1,\ 2$$

which implies that at critical points

$$\mathbf{n}\,(x_1,\ x_2) = \pm\,(\theta-\theta_0)\,/\,|\,\theta-\theta_0\,| \tag{2.1}$$

where the plus sign corresponds to the critical point in the illuminated region and the minus sign to that lying in the shade. These are simple points, since the determinant of the matrix of second derivatives of the phase with respect to coordinates $x_1$ and $x_2$ is nonzero.

Critical points of $f_2\,(\theta)$ are determined by conditions

$$\partial\tau\,/\,\partial y = (\theta,\ \partial\mathbf{r}\,/\,\partial y), \quad \partial\tau\,/\,\partial z = (\theta,\ \partial\mathbf{r}\,/\,\partial z)$$

from which we obtain that at these points

$$\partial\mathbf{r}\,/\,\partial y = \theta \tag{2.2}$$

Critical points of this type are no longer simple, since the matrix of the phase second derivatives is of rank unity.

We shall now determine the asymptotics of integrals for $|\theta-\theta_0| \geqslant \mu^{-1/3+\varepsilon}$ which we assume to be satisfied for some small $\varepsilon > 0$. It follows from (2.1) and (2.2) that it is possible to select the subdivision of unity so that the critical points of the integral $f_2(\theta)$ lie outside the carrier, and that at critical points of integrals $f_1(\theta)$ and $f_3(\theta)$ $\chi_1$ and $\chi_3$ are equal unity. Note that since $(\theta+\theta_0, \theta-\theta_0) = 0$, the integrand of the last integral in (1.3) at a stationary point is zero. Hence $f_3(\theta) = O(\mu^{-2/3})$, and by virtue of the selected subdivision of unity $f_2(\theta)$ is exponentially small. Applying to the first integral in (1.3) the method of stationary phase [4] we obtain

$$f(\theta) = \varkappa^\circ \exp\{-ik(\theta-\theta_0, \mathbf{r}^\circ)\}(1 + O[(\mu|\theta-\theta_0|)^{-1}]) \qquad (2.3)$$

$$\varkappa^\circ = \frac{i\exp(i\pi/2)\,h_1^\circ h_2^\circ}{2(|\det(\theta-\theta_0, \partial^2\mathbf{r}^\circ/\partial x_i\,\partial x_j)|)^{1/2}}\,\frac{1-A^\circ}{1+A^\circ}(\theta-\theta_0, \mathbf{n}^\circ)$$

where $U^\circ$ is equal to the value of $U$ $(U = h_1, h_2, \mathbf{r}, A, \mathbf{n})$ at point $(x_1^\circ, x_2^\circ)$ for which $\mathbf{n}(x_1^\circ, x_2^\circ) = (\theta-\theta_0)/|\theta-\theta_0|$. It follows from (2.3) that

$$f(\theta) \sim -\frac{1}{2}(K^\circ)^{-1/2}\frac{1-A^\circ}{1+A^\circ}\exp\{-ik(\theta-\theta_0, \mathbf{r}^\circ)\} \qquad (2.4)$$

where $K$ is the Gaussian curvature. Formula (2.4) is in agreement with the results in [5].

3. **Asymptotics of the pattern of scatter for** $\mu^{-1} \ll |\theta-\theta_0| \leqslant \mu^{-1/3+\varepsilon}$. By a suitable selection of the subdivision of unity we attain that the critical points of the first and third integrals in (1.3) lie outside the carriers $\chi_1$ and $\chi_3$, from which follows the exponential smallness of these integrals. The method of stationary phase cannot be applied to integral $f_2(\theta)$, since the critical points that lie inside the carrier $\chi_2$ are not simple.

There are two critical points of the second integral in (1.3) which satisfy condition (2.2), one of which $(y_1, z_1)$ is in the illuminated region $(y_1 < 0)$, and the second $(y_2, z_2)$ in the shade region $(y_2 > 0)$. We represent $\chi_2$ in the form $\chi_{21} + \chi_{22}$, where $\chi_{2i} \in C^\infty$, $0 \leqslant \chi_{2i} \leqslant 1$ and the carrier $\chi_{2i}$ includes the band of geodesics that contain the single critical point $(y_i, z_i)$. Function $f_2(\theta)$ is, respectively, of the form $f_{21}(\theta) + f_{22}(\theta)$. For the integral $f_{2i}(\theta)$ we express $\tau - (\theta, \mathbf{r})$ in the form

$$\tau(y, z) - (\theta, \mathbf{r}(y, z)) = \Phi_i(z) + \left(\theta, \frac{\partial\mathbf{r}(y_i, z_i)}{\partial y}\right)(y-y_i) -$$

$$\sum_{n=1}^{\infty}\left(\theta, \frac{\partial^n\mathbf{r}(y_i, z)}{\partial y^n}\right)\frac{(y-y_i)^n}{n!}$$

$$\Phi_i(z) = \tau(y_i, z) - (\theta, \mathbf{r}(y_i, z))$$

Since $|y|, |y_i| = O(\mu^{-1/3+\varepsilon})$, hence for any $\varepsilon > 0$

$$k\,|\tau(y, z) - (\theta, \mathbf{r}(y, z)) - \Phi_i(z)| = O(\mu^{3\varepsilon})$$

Thus the rapidly oscillating terms in $f_{2i}(\theta)$ depend only on $z$ and have each a single stationary point inside the carrier. Consequently, the method of stationary phase can be applied to the integral with respect to $z$. According to that method from (1.3) we obtain

$$f_2(\theta) \sim \sum_{m=1}^{2} \Omega_m(\theta) \exp\{ik(\tau_m - (\theta, \mathbf{r}_m))\} N_m \tag{3.1}$$

$$\Omega_m = \frac{k^{-1/6}(g_m)^{1/2} \exp(\pm i\pi/4)}{2^{1/3}\pi \,|\,\partial t_m^2/\partial y\,|^{1/2}}, \quad t_m^2 = \left(\frac{\partial \mathbf{r}_m}{\partial z}, \frac{\partial \mathbf{r}_m}{\partial z}\right)$$

$$N_m = \int_l \int_{-\mu^\varepsilon}^{\mu^\varepsilon} \chi_{2m} \exp\left\{is\zeta + \frac{i}{3}(s - s_m)^3\right\} \sigma_m(s - s_m, \zeta)\, d\zeta\, ds$$

$$\sigma_m(x, \zeta) = [w_1'(\zeta) - ixw_1(\zeta)]\, G_m(\zeta)$$

where $g$ is the determinant of the metric tensor of the surface defined in coordinates $(y, z)$, and the subscript $m$ indicates that the respective quantity is calculated at point $(y_m, z_m)$.

Since the integral in (3.1) contains the exponent of the third power polynomial in $s$, its asymptotics must contain integrals with respect to $\zeta$ of Airy functions and their derivatives.

For $\zeta \in l_2 \cup l_3$ we have the identity

$$\sigma_m(x, \zeta) = \sigma_m^{(1)}(x, \zeta) + \sigma_m^{(2)}(x, \zeta)$$

$$\sigma_m^{(1)} = -v'(\zeta) + ixv(\zeta)$$

$$\sigma_m^{(2)} = (1 + iC_m x) / [w_1(\zeta) + C_m w_1'(\zeta)]$$

Then the integral $N_m$ is of the form $N_m = I_m^{(1)} + I_m^{(2)}$, where

$$I_m^{(n)} = \int_l \int_{-\mu^\varepsilon}^{\mu^\varepsilon} \chi_{2m} \exp\left\{is\zeta + \frac{i}{3}(s - s_m)^3\right\} \sigma_m^{(n)}(s - s_m, \zeta)\, d\zeta\, ds$$

The integral $I_m^{(1)}$ after integration with respect to $\zeta$ followed by integration by parts with respect to $s$, assumes the form

$$I_m^{(1)} = -\frac{\sqrt{\pi}}{s_m} \int_{-\mu^\varepsilon}^{\mu^\varepsilon} \frac{d\chi_{2m}}{ds} \exp\left\{-iss_m(s - s_m) - \frac{i}{3} s_m^3\right\} ds$$

This integral can be made small, $\sim \mu^{-1/4}$ by suitable selection of the subdivision of unity.

Let us investigate the integral $I_m^{(2)}$. With $s_m > 0$ we extend the interval $(-\mu^\varepsilon, \mu^\varepsilon)$ up to the contour $l_-(-\mu^\varepsilon, 5\pi/6) \cup [-\mu^\varepsilon, \mu^\varepsilon] \cup l_+(\mu^\varepsilon, \pi/6)$ and, then, distort the latter into $l_-(s_m, 5\pi/6) \cup l_+(s_m, \pi/6)$. Similarly, with $s_m < 0$ we transform the interval $(-\mu^\varepsilon, \mu^\varepsilon)$ into the contour $l_-(s_m, 3\pi/2) \cup l_+(s_m, \pi/6)$.

We replace integration of integral $I_m^{(2)}$ over the interval $(-\mu^\varepsilon, \mu^\varepsilon)$ by integration over the corresponding contours. The ensuing change of such integral is exponentially small with respect to $\mu$. The integral with respect to $s$ is then expressed in terms of the Airy function and its derivatives. From this we obtain the asymptotics for $N_m$

$$N_m \sim 2\sqrt{\pi} M_m = 2\sqrt{\pi}\left\{\frac{i}{2}\int_{l_2} e^{is_m\zeta} R_1^{(1)}(\zeta)\,d\zeta + \int_{l_3} e^{is_m\zeta} R_2^{(1)}(\zeta)\,d\zeta - \frac{1}{2s_m}\right\}$$

$$R_1^{(1)} = \frac{w_2 + C_m w_2'}{w_1 + C_m w_1'}, \quad R_2^{(1)} = \frac{v + C_m v'}{w_1 + C_m w_1'}$$

Since

$$\tau_m \sim (\theta_0, \mathbf{r}_m) + k^{-1} s_m^3 / 3 \qquad (3.2)$$

$$\tfrac{1}{2}\partial t_m^2 / \partial y \sim -K_m s_m g_m (2R_m / k)^{1/3}$$

where $K_m$ is the Gaussian curvature at point $(y_m, z_m)$, we finally obtain for the pattern of scatter for $\mu^{-1} \ll |\theta - \theta_0| \leqslant \mu^{-1/3+\varepsilon}$ an expression of the form

$$f(\theta) \sim \sum_{m=1}^{2} |K_m \pi s_m|^{-1/2} \exp\left\{-ik(\theta - \theta_0, \mathbf{r}_m) + \frac{i}{3} s_m^3 \pm \frac{i\pi}{4}\right\} M_m \qquad (3.3)$$

4. The pattern of scatter for $|\theta - \theta_0| \lesssim \mu^{-1}$. It follows from Sects. 2 and 3 that the large parameter in the integrals is the quantity $|\theta - \theta_0|\mu$. Hence in that range of $\theta$ the integrands cannot be assumed as rapidly oscillating, and consequently $f_2(\theta) = O(\mu^{-1/3+\varepsilon})$, $(\theta - \theta_0, \mathbf{n}) = O(\mu^{-1})$, and $f_1(\theta) = O(1)$.

Substituting in the third of formulas (1.3) $\theta_0$ for $\theta$ we obtain

$$f(\theta) \sim f_3(\theta) \sim \frac{ik}{2\pi}\int_S \chi_3(\theta_0, \mathbf{n}) \exp\{-ik(\theta - \theta_0, \mathbf{r})\}\,ds \qquad (4.1)$$

Further simplifications are only possible when $|\theta - \theta_0|\mu \ll 1$. Substituting in the integral (4.1) unity for the exponent, we obtain an integral over the shaded part of the surface which is equal to the area of the shell $S$ projection in the direction $\theta_0$, i.e.

$$f(\theta) \sim ikS / (2\pi) \qquad (4.2)$$

Note that formulas (2.4), (3.3), (4.1) and (4.2) for the pattern of scatter agree with each other. Formula (2.4) which is valid for large angles between the direction of the incident field and that of the observer, also follows directly from the radiation method. At small angles radiation formulas are no longer valid and the solution becomes more complex. Note that at small angles $\theta$ and $\theta_0$ the Fresnel integrals which appear in [formulas for] the penumbra at finite distances from the surface do not occur.

Remark. The same result can be obtained for the Dirichlet and Neumann problems on the exterior of the convex region in $R^3$. The respective formulas for the Neumann and Dirichlet problems are obtained from (2.4) and (3.3) by the formal substitution of $A \equiv \infty$ and $C \equiv \infty$, and $A \equiv 0$ and $C \equiv 0$, respectively. Proof of the obtained formulas is similar to that given in [6].

Note that similar results were obtained in [7] for the problem of scatter of a plane high-frequency electromagnetic wave over a perfectly conducting circular cylinder by analyzing the exact solution.

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Translated by J. J. D.